Levi form of $M \subseteq \mathbb{C}^{n+1}$ in terms of ρ.

Let $\theta = i\,\iota^*\partial\rho$.

<u>Cartan's Formula</u>. For a 1-form ω,

$$2d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

Thus, applying this to def. of $\mathcal{L}^\theta_p$, we get

$$2i\,\mathcal{L}^\theta_p(Z_p, W_p) = \theta([Z,\overline{W}]) = \underbrace{\overline{W}\theta(Z) - Z\theta(\overline{W})}_{\substack{\text{both } = 0 \text{ since} \\ \theta^\perp = T^{1,0} + T^{0,1}}}$$

$$-2d\theta(Z,\overline{W}) = -2d\theta(Z,\overline{W}).$$

If we write $Z = \sum \xi^j \frac{\partial}{\partial z_j}$, $W = \sum \eta^j \frac{\partial}{\partial z_j}$ (in $\mathbb{C}^{n+1}$), then

$$d\theta(Z,\overline{W}) = i\,d\partial\rho(Z,\overline{W}) = i\,\overline{\partial}\partial\rho(Z,\overline{W})$$

$$= -i\sum_{j,k} \frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}\,\xi^j\,\overline{\eta^j}$$

<u>Prop</u>: In the notation above,

$$\mathcal{L}^\theta_p(Z_p,W_p) = \frac{1}{2i}\theta([Z,\overline{W}]) = i\,d\theta(Z,\overline{W}) = \sum_{j,k} \frac{\partial^2\rho(p)}{\partial z_j\partial\overline{z}_k}\,\xi^j\,\overline{\eta^j}.$$

Suppose $Z_1, \dots, Z_n$ is a local frame for $T^{1,0}M$,

$$Z_\alpha = \sum_{k=1}^{n+1} \xi_\alpha^k \frac{\partial}{\partial z_k}, \quad \alpha = 1, \dots, n.$$

Then, we can identify the Levi form at $p \in M$ with a Hermitian $n \times n$ matrix

$$g_{\alpha\bar\beta} = \mathcal{L}_p^\theta(Z_\alpha(p), \overline{Z_\beta(p)}) = \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar z_k}(p)\, \xi_\alpha^j(p) \overline{\xi_\beta^k(p)}$$

If we write $H_\rho = \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar z_k}(p) \right)_{j,k}$ and ← $(n+1) \times (n+1)$.

$$\xi = \begin{pmatrix} - \xi_1 - \\ \vdots \\ - \xi_n - \end{pmatrix}$$ ← $n \times (n+1)$ matrix

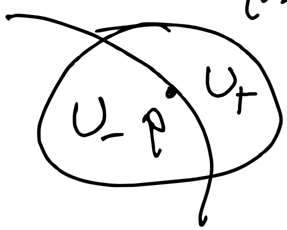
then the Hermitian $n \times n$ matrix

$$(g_{\alpha\bar\beta}) = \xi H_\rho \xi^*$$

represents the Levi form $\mathcal{L}_p^\theta$ for $\theta = i e^* \partial\rho$.

Recall M is strictly ψcvx at p if the Levi form is definite at p. Replacing ρ by $-\rho$ if needed, WLOG, M is <u>s. ψcvx</u> at p $\Leftrightarrow$ $(g_{\alpha\bar\beta})$ is <u>positive definite</u> (for some ρ). If U is small open nbhd of $p \in M$, then

$$U = \underset{\rho>0}{U_+} \cup \underset{\rho=0}{M} \cup \underset{\rho>0}{U_-}$$



pos. semidef

Def. M is <u>ψcvx</u> in U if $(g_{\alpha\bar\beta}) \geq 0$.
In this case, U_- is called the <u>ψcvx side</u> of M.

Rem. Cf. $\Omega \subseteq \mathbb{C}^{n+1}$ domain, $M = \partial\Omega$ smooth,
Ω ψcvx $\Leftrightarrow$ M is ψcvx in U_p, $\forall p \in M$
and $\Omega \cap U_-$ is ψcvx side of M.

Holomorphic extension of CR functions on $M \subseteq \mathbb{C}^{n+1}$.

Recall that the notion of a CR function $u: M \to \mathbb{C}$ was modeled on restrictions of holomorphic functions: u is CR $\Leftrightarrow$ $\bar{Z}u = 0$, $\forall$ sections Z of $\mathcal{V} = T^{1,0}M$. What about the converse? "Globally", this is Hartogs:

<u>Hartogs Thm - CR version</u>. Let $\Omega \subseteq \mathbb{C}^{n+1}$ be bdd domain with smooth connected bdry $M = \partial\Omega$. If u is a C^k- CR function on M, then $\exists\, U \in \mathcal{O}(\Omega) \cap C^k(\bar{\Omega})$ s.t. $u = U|_M$.

<u>Pf</u>. In the original HT, u already came from some holomorphic function in $\Omega \setminus K$, $K \subset\subset \Omega$. The proof here is similar. First extend u to $u_1 \in C^k(\bar{\Omega})$ s.t. $u_1|_M = u$. This is clearly doable. Then, improve u_1 to $u_2 \in C^k(\bar{\Omega})$ s.t. $u_2|_M = u$ and $\bar{\partial} u_2 = O(\rho^2)$ $(M = \{\rho = 0\})$.

If we can do this, then the $(0,1)$-form $f = -\bar\partial u_2$ in Ω, $f=0$ in Ω^c is C^1, has compact support and satisfies $\bar\partial f = 0$. Proceeding as in the original HT, correcting u_2 by solving $\bar\partial v = f$ finishes the proof.

How to find u_2? Outline only (see [Hör, §2.3] for details).

Step 1. $u_1|_M = u$ and u CR $\Rightarrow$ ($\Leftrightarrow$ actually)

$$\bar\partial u_1 \wedge \bar\partial\rho = 0 \text{ on } M.$$

Step 2. Step 1 $\Rightarrow$ $\bar\partial u_1 = h_0 \bar\partial\rho + h_1\rho$ $\Rightarrow$ (h_1: $(0,1)$-form)

$$\bar\partial(u_1 - h_0\rho) = (h_1 - \bar\partial h_0)\rho = h_2\rho$$

Thus, $0 = \bar\partial(h_2\rho) = (\bar\partial h_2)\rho + \bar\partial\rho \wedge h_2 \Rightarrow$

$\bar\partial\rho \wedge h_2 = 0$ on M $\Rightarrow$

$$h_2 = h_3\bar\partial\rho + h_4\rho$$

$\Rightarrow$ Let $u_2 = u_1 - h_0\rho - h_3\frac{\rho^2}{2}$. Check $\bar\partial u_2 = O(\rho^2)$!